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CONTROL OF THE SURFACE OF A SECTIONED BIMORPH PLATE

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At the present time, optical elements with controllable characteristics are widely used in various fields of engineering [1]. Among these systems of adaptable optics are bimorph mirrors made of piezoelectric ceramics. The deformations of bimorph rectangular mirrors were studied in [2–4].

The present study deals with a circular mirror consisting of a piezoelectric ceramic layer coupled to a metal layer with a reflecting vaporized coating. Henceforth, this kind of mirror is called a semipassive bimorph. The piezoelectric element is covered by a thin sectionalized electrode coating. To change the mirror shape, control voltages are supplied to various sections. Here the problem of optimum control arises to determine values of the voltages that would make it possible to approximate the specified shape of the mirror with sufficient effectiveness. A similar statement of the problem was considered in [5], but in terms of a less strict approach.

1. We consider a bimorph plate consisting of a piezoelectric ceramics layer with a thickness of h_1 bonded to a metal layer with a thickness of h_2. We refer the plate to a Cartesian reference system $Ox_1x_2x_3$ as shown in Fig. 1. We assume that the deformation of the plate occurs owing to the voltage $v(x_1, x_2)$ applied to an infinitely thin electrode covering the boundary $x_3 = h_1$. In addition, the electrode covering the boundary $x_3 = 0$ is shorted. The results of the asymptotic analysis of equilibrium of an inhomogeneous electro-elastic slab carried out in [6] indicate the possibility of using the Kirchhoff hypotheses here, and similarly to [7] one obtains the following expression for the bending moments M_{ij}:

$$M_{11} = -D(w_{,11} + \nu_0 w_{,22}) - \alpha v, \quad M_{22} = -D(\nu_0 w_{,11} + w_{,22}) - \alpha v, \quad M_{12} = -D(1-\nu_0) w_{,12},$$

where the effective flexural rigidity D, Poisson's ratio ν_0, and the control coefficient α are defined by the following formulas:

$$D = \frac{1}{3}(h_1^3 B_{11}(1+\beta) + h_2^3 A_{11}) - \frac{1}{8}(B_{22} + A_{22}), \quad \nu_0 = \frac{D_{22}}{D},$$

$$D_{22} = \frac{1}{3}(h_1^3 B_{11}(\nu+\beta) + \nu_m h_2^3 A_{11}) + \frac{1}{8}(B_{22} - A_{22}),$$

$$\alpha = -\frac{1}{2} d\left(h_1 - \frac{B_{11}h_1^2(1+\nu) - A_{11}h_2^2(1+\nu_m)}{B_{11}h_1(1+\nu) + A_{11}h_2(1+\nu_m)}\right),$$

$$A_{22} = \frac{(B_{11}h_1^2(1+\nu) - A_{11}h_2^2(1+\nu_m))^2}{B_{11}h_1(1+\nu) + A_{11}h_2(1+\nu_m)},$$

$$B_{22} = \frac{(B_{11}h_1^2(1-\nu) - A_{11}h_2^2(1-\nu_m))^2}{B_{11}h_1(1-\nu) + A_{11}h_2(1-\nu_m)},$$

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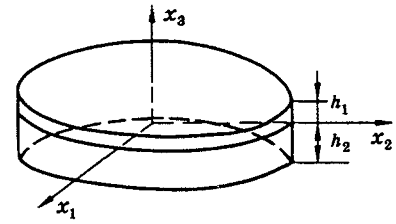

Fig. 1

$$d = \frac{d_{31}}{s_{11}(1-\nu)}, \quad A_{11} = \frac{1}{s_{11}^{\mathrm{m}}(1-\nu_{\mathrm{m}}^2)}, \quad B_{11} = \frac{1}{s_{11}(1-\nu^2)},$$
$$\beta = \frac{d^2}{4\text{Э} B_{11}}, \quad \nu_{\mathrm{m}} = -\frac{s_{12}^{\mathrm{m}}}{s_{11}^{\mathrm{m}}}, \quad \nu = -\frac{s_{12}}{s_{11}}, \quad \text{Э} = \text{Э}_{33} - 2d_{31}d.$$

Here s_{ij} and s_{ij}^{m} are the elastic flexibilities of the ceramics and the metal respectively, P_{33} is the permittivity, d_{31} is the piezoelectric coefficient, and $w = w(x_1, x_2)$ is the deflection of the plate surface $x_3 = 0$. Using the equilibrium equation of the plate

$$M_{11,11} + 2M_{12,12} + M_{22,22} = 0,$$

we obtain the equation of bending of a semipassive bimorph,

$$D\Delta^2 w = -\alpha\Delta v, \tag{1.1}$$

which has the same form as the classical equation of bending of thin elastic plates with effective flexural rigidity D. Here, the expression $-\alpha\Delta v$ appears as transverse load q.

The boundary conditions at the plate contour Γ for Eq. (1.1) are defined by the physical supporting conditions and the kinematic Kirchhoff hypotheses [6] and have the same form as the known boundary conditions for bending of elastic plates. Namely, for the fixed-end conditions $w = \partial w/\partial n = 0$, at Γ for the hinged support $w = 0$ and $M_{nn} = 0$, and, finally, for the free edge

$$M_{nn} = 0, \quad Q_n^* = Q + \frac{\partial M_{n\tau}}{\partial \tau} = 0, \quad Q_n = -D\frac{\partial}{\partial n}(\Delta w) - \alpha\frac{\partial v}{\partial n} = 0, \tag{1.2}$$

where n is the unit vector normal to Γ, τ is the unit tangent vector, and Q_n is the transverse shear force.

2. We consider a circular bimorph plate of radius a with a free edge. In this case it is convenient to study the boundary-value problem of bending using the polar coordinates ρ and θ ($x_1 = a\rho\cos\theta$, $x_2 = a\rho\sin\theta$, and $0 \leqslant \rho \leqslant 1$).

We write the boundary-value problem (1.1) and (1.2) in dimensionless form:

$$\Delta^2 w = \Delta U, \quad k_1 w = U, \ \rho = 1, \quad k_2 w = U_{,\rho}\,, \ \rho = 1. \tag{2.1}$$

Here Δ is the Laplacian operator and the differential expressions k_1 and k_2 are defined in the polar coordinate system by the relations

$$k_1 = \frac{\partial^2}{\partial\rho^2} + \nu_0\Big(\frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial}{\partial\theta^2}\Big),$$
$$k_2 = \frac{\partial}{\partial\rho}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\Big((2-\nu_0)\frac{\partial}{\partial\rho} - (3-\nu_0)\frac{1}{\rho}\Big), \tag{2.2}$$
$$U = -\frac{\alpha a^2}{Dw_0}v, \quad W = \frac{w}{w_0}$$

(w_0 is a characteristic deflection of the plate).

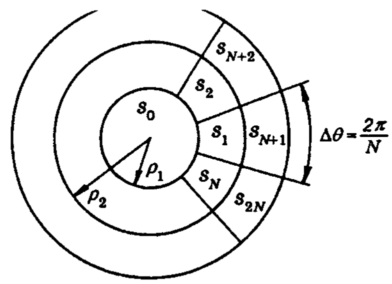

Fig. 2

We divide the circle $\rho \leqslant 1$ into $2N+1$ regions (sections) by the circumferences $\rho = \rho_1$ and $\rho = \rho_2$ $(0 < \rho_1 < \rho_2 < 1)$ and the segments of the rays $\theta = k\Delta\theta$ $(\Delta\theta = 2\pi/N,\ k = 0, 1, \dots, N-1)$ with $\rho_1 \leqslant \rho \leqslant 1$. As a result, the plate will be divided into a circular region, s_0 and $2N$ sector regions s_n, $n = 1, 2, \dots, 2N$ (Fig. 2). Let us construct the solution to problem (2.1) and (2.2) for the piecewise constant function U, which corresponds to the above sectioning of the plate.

We assume the value of the function U to be constant in each of the regions s_k:

$$U(\rho,\theta) = \sum_{k=0}^{2N} U_k \chi(s_k)$$

(U_k is a constant value of U in the region s_k and $\chi(s_k)$ is the characteristic function of the region s_k).

Using a Fourier series expansion of the unknown function W with respect to θ, one obtains the expression

$$W(\rho,\theta) = \sum_{k=-3}^{2N} U_k W_k(\rho,\theta), \tag{2.3}$$

where

$$U_{-3} = A_0,\ U_{-2} = A_1^{(1)},\ U_{-1} = A_1^{(2)},\ W_{-3} = 1,\ W_{-2} = \rho\cos\theta,\ W_{-1} = \rho\sin\theta,$$

$$W_0 = \frac{1}{2}\gamma_1\rho_1^2\rho^2 + \begin{cases} \frac{1}{2}\rho_1^2\left(\ln\rho_1 - \frac{1}{2}\right) + \frac{1}{4}\rho^2, & \rho \leqslant \rho_1, \\ \frac{1}{2}\rho_1^2 \ln\rho, & \rho \geqslant \rho_1, \end{cases}$$

$$W_k(\rho,\theta) = W_0^{(1)}(\rho) + \sum_{n=1}^{\infty} W_n^{(1)}(\rho) T_{kn}(\theta) \quad (1 \leqslant k \leqslant N), \tag{2.4}$$

$$W_k(\rho,\theta) = W_0^{(2)}(\rho) + \sum_{n=1}^{\infty} W_n^{(2)}(\rho) T_{kn}(\theta) \quad (N < k \leqslant 2N),$$

$$T_{kn}(\theta) = \cos n\theta S_{kn} + \sin n\theta C_{kn}, \quad S_{kn} = \sin n\theta_k - \sin n\theta_{k-1},$$

$$C_{kn} = \cos n\theta_{k-1} - \cos n\theta_k, \quad \theta = k\Delta\theta = k\frac{2\pi}{N}.$$

We note that $W_n^{(k)}(\rho)$ are the known functions of ρ; because of their cumbersome form they are not given here.

3. The obtained solution $W(\rho,\theta)$ to the bending problem may be considered as a function depending on the U_k value, i.e.,

$$W = W(\rho, \theta, U_{-3}, U_{-2}, \dots, U_{2N}).$$

In this case, the dependence of the deflection W on U_k defined by (2.3) makes it possible to investigate the practically important problem of optimization of the shape of a semipassive bimorph.

TABLE 1

n	k					
	4	5	6	7	8	11
	$J_k \cdot 10^3$					
	0	5.1	5.1	0.611	0.611	0.32
	U_n					
−3	0.32	0	0	0	0	−0.78
−2	0	0	0	0	2.76	0
−1	0	0	0	2.76	0	0
0	0.53	0	0	0	0	−26.5
1	0.53	5.1	5.62	5.62	13.57	−0.2
2	0.53	5.1	−5.1	13.57	5.62	−0.2
3	0.53	−5.1	−5.1	13.57	−5.62	−0.2
4	0.53	−5.1	5.1	5.62	13.57	−0.2
5	0.53	5.1	5.1	−5.62	−13.57	−0.2
6	0.53	5.1	−5.1	−13.57	−5.62	−0.2
7	0.53	−5.1	−5.1	−13.57	5.62	−0.2
8	0.53	−5.1	5.1	−5.62	13.57	−0.2
9	0.53	−11.24	−11.24	5.66	13.66	36.0
10	0.53	−11.24	11.24	13.66	5.66	36.0
11	0.53	11.24	11.24	13.66	−5.66	36.0
12	0.53	11.24	−11.24	5.66	−13.66	36.0
13	0.53	−11.24	−11.24	−5.66	−13.66	36.0
14	0.53	−11.24	11.24	−13.66	−5.66	36.0
15	0.53	11.24	11.24	−13.66	5.66	36.0
16	0.53	11.24	−11.24	−5.66	13.66	36.0

Suppose that it is required to provide the deflection form of a bimorph plate close to a certain specified continuously differentiable function $F(\rho, \theta)$ by choosing the control voltages U_k. We expand this function into a generalized Fourier series in a system of Zernike trigonometric polynomials $Z_k(\rho, \theta)$ [8] forming the basis in $L_2(S)$:

$$Z_1 = 1, \quad Z_2 = 2\rho\cos\theta, \quad Z_3 = 2\rho\sin\theta, \quad Z_4 = 2\rho^2 - 1, \quad Z_5 = \rho^2 \sin 2\theta, \ldots .$$

The expansion has the form

$$F(\rho, \theta) = w_0 \sum_{k=1}^{\infty} a_k Z_k(\rho, \theta).$$

Next, we construct the functional

$$J(U_{-3}, U_{-2}, \ldots, U_{2N}) = \frac{1}{S} \int_S \left(W - \sum_{k=1}^{\infty} a_k Z_k \right)^2 dS, \tag{3.1}$$

defining the deviation of the deflection W from the function F (S being the area of the plate and $W = \Sigma a_k W_k$).

The values of U_m corresponding to the minimum value of functional (3.1) will be called the optimal values. Using the superposition principle, we express the control voltages U_m in the form

$$U_m = \sum_{k=1}^{\infty} a_k U_{km},$$

where U_{km} is the control voltage at the mth electrode which approximates the kth Zernike polynomial and is determined from the following algebraic system:

$$\sum_{n=-3}^{2N} U_{kn} B_{nm} = f_{km}, \quad B_{nm} = \int_S W_n W_m \, dS, \quad f_{km} = \int_S Z_k W_m \, dS. \tag{3.2}$$

In addition, U_{km} corresponds to the minimum value of the kth residual functional

$$J_k = \frac{1}{S}\Big(\int_S Z_k^2 dS - \sum_{m=-3}^{2N} U_{mk} f_{km}\Big),$$

characterizing the mean-square deviation of the form of the plate surface from the kth Zernike polynomial. Using the minimum condition, we obtain the following linear algebraic system of equations:

$$\sum_{n=-3}^{2N} U_n \int_S W_n W_m dS = \sum_{k=1}^{\infty} a_k \int_S Z_k W_m dS, \quad m = -3, -2, \ldots, 2N.$$

System (3.2) posseses a number of effective computational properties, i.e., the matrix B_{nm} is symmetrical and non-negative definite, and does not depend on the form of the function F. It should also be noted that the integrals appearing in (3.2) are easy to calculate analytically with respect to the angle θ using formulas (2.4), so that the coefficients of the matrix and the components of the vector on the right-hand side of the system are expressed only in terms of the integrals with respect to the radial variable ρ.

Optimization defining the dimensionless sectioning radii ρ_1 and ρ_2 for the basic 17-electrode coating is carried out. The values $\rho_1 = 0.4$ and $\rho_2 = 0.7$ are found and the control voltages U_n $(n = -3, -2, \ldots, 16)$ are calculated which are listed in Table 1, where the translational displacement of the plate U_{-3} and the rotations of the plate U_{-2} and U_{-1} about the coordinate axes correspond to the values $n = -3, -2, -1$. Here, k is the Zernike polynomial number and n is the electrode number. We give in Table 1 the data starting with $k = 4$ only, since the polynomials Z_k for $k = 1$, 2, and 3 correspond to the rigid body displacements of the plate.

Note that the results for $k = 9$ and $k = 10$ are not given in Table 1, since the corresponding Zernike polynomials proportional to sin 3θ and cos 3θ cannot be approximated satisfactorily for the chosen sectioning ($N = 8$). To obtain a good approximation of these polynomials, one should take $N = 6$, 9, or 12.

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